Note on the Kadison-Singer Problem and its Solution

Raghav Somani

March 21, 2021

The Kadison-Singer problem arose from the work on quantum mechanics done by Paul Dirac in the 1930s. It was formalized by Kadison and Singer in 1959 as a problem in functional analysis [KS59] while trying to make Dirac's axioms for quantum mechanics mathematically rigorous in the context of von Neumann algebras (Definition 26). The problem is equivalent to fundamental problems in areas like Operator theory, Hilbert and Banach space theory, Frame theory, Harmonic Analysis, Discrepancy theory, Graph theory, Signal Processing and theoretical Computer Science. The problem in its original form is considered as a basic question about the most fundamental C^*-algebra (Definition 25) thereby generating a fairly substantial literature over the past few decades. The problem in its original form is the below question.

Question 1 (Kadison-Singer problem)**.** Does every pure state on the maximal diagonal subalgebra $D(H) \subset B(H)$ have a unique extension to a state in $B(H)$, where H is a separable Hilbert space, and $B(H)$ is the von Neumann algebra of bounded linear operators on H?

Refer to the chain of definitions in §A for completeness. Due to its ubiquitous nature across mathematics, Question 1 has been shown to be equivalent to a number of conjectures, some of which include the Anderson's paving conjecture [And79a, And79b, And81], and the Weaver's discrepancy theoretic conjecture [Wea04].

The Kadison-Singer problem had been long standing and defied the efforts of most Mathematicians until it was recently solved by Adam Wade Marcus, Daniel Alan Spielman and Nikhil Srivastava [MSS15, MSS14, MS17] for which they were awarded the George Pólya Prize in Mathematics in 2014, and very recently, the Michael and Sheila Held Prize in 2021. The authors solve the Kadison-Singer problem by proving the Anderson's Paving conjecture and the Weaver's discrepancy theoretic conjecture which we will state and describe in later sections. The proof uses an existence argument which reduces the problem to bounding the roots of the expected characteristic polynomial of certain random matrices employing tools from the theory of random polynomials.

For a layman discussion on the history, the consequence, and the solution of the problem, readers are highly encouraged to read the interesting Quanta Magazine article [Kla15] before continuing.

1 Introduction and History

From the axioms of quantum mechanics, a quantum system consists of observables and states. An observable is a physical quantity which we would like to measure, and is represented via a self-adjoint operator on a complex separable Hilbert space H, for example $\ell_2\mathbb{N}$. A state is represented by a vector in this Hilbert space. The question now is, what is a good basis for the representation of these states? Dirac prescribed a way to do this which is to first find a maximal family of commuting observables which can be measured simultaneously (no uncertainty principle holding us back), then the required basis could be set to the common eigenbasis of this maximal family of these commuting observables. Any state could then be represented in this basis and would be its representation. This seemed fine unless von Neumann raised some mathematical objections to this solution. First, we cannot simply diagonalize an infinite matrix and get the eigenvectors, and second, Dirac was able to get around with the first objection with the help of delta functions, but delta functions are not vectors in a Hilbert space at the first place. Motivated by these problem, von Neumann introduced a different mathematical framework of operator algebras for quantum mechanics, where observables are again self-adjoint bounded operators on a separable Hilbert space H, which generate the algebra of all bounded operators on H, denoted as $B(H)$. The notion of commuting observables is also the same where the operators commute and we obtain an abelian C^*-subalgebra $\mathcal{A}$ of $B(H)$. However, a state on the C^*-algebra $\mathcal{A}$ is now a linear functional $\rho\colon \mathcal{A} \to \mathbb{C}$ on a C^*-subalgebra $\mathcal{A} \subset B(H)$ which maps the identity to 1 and is non-negative for observables of the form M^*M for $M \in \mathcal{A}$. A class of examples are the set of 'vector states' defined for a unit vector $v \in H$ as $\rho(A) = \langle v, Av \rangle$. It can be verified that a convex combination of states is again a state, and the set of states is a compact set under the w^*-topology (Definition 27). Therefore, from the Krein-Milman theorem, the set of states can be represented as a convex hull of the set of extreme points, which are called as 'pure states'. In other words, a pure state cannot be represented as a convex combination of any two different states. Two properties of states $\rho\colon \mathcal{A} \to \mathbb{C}$ that we shall use from hereon are:

1. Cauchy-Schwartz: $|\rho(MN)|^2 \leq \rho(M^*M)\rho(N^*N)\ \forall\ M, N \in \mathcal{A}$, and
2. $\rho(M) \leq \|M\|_2\ \forall\ M \in \mathcal{A}$.

Kadison and Singer looked at the problem is the 1950s and wanted to understand if Dirac's procedure makes sense in the von Neumann formulation of quantum mechanics. That would mean to first find a maximal set of commuting observables creating a maximal abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and then consider pure states of this subalgebra since there need not be a notion of eigenstates anymore. A natural question therefore comes up is that if we have a pure state on the C^*-subalgebra $\mathcal{A}$ of commuting observables, then is it completely determined by the values it takes on this subalgebra? In mathematical terms, does each state defined on the C^*-subalgebra $\mathcal{A}$ have a unique extension to all of $B(H)$? The vector pure states can be shown to be completely determined by their values on $\mathcal{A}$, but there are substantially more pure states in $\mathcal{A}$ if H is infinite dimensional. Since the set of states is a convex compact set, it is sufficient to ask this question only for pure states. Kadison and Singer asked this question [KS59] and showed that the answer in general is no, by constructing an abelian subalgebra where a pure state defined on this subalgebra had two different extensions to $B(H)$. However, they could not settle it for the case when $\mathcal{A}$ is the maximal abelian C^*-subalgebra of bounded diagonal linear operators $D(H) \subset B(H)$, which is precisely the problem in Question 1 for $H = \ell_2\mathbb{N}$.

The existence of an extension can be constructed simply by ignoring the off-diagonal elements of the representation of the linear operator, i.e., given a pure state $\rho\colon D(H) \to \mathbb{C}$, we can define its extension $\rho_0\colon B(H) \to \mathbb{C}$ as $\rho_0(A) = \rho(\mathrm{diag}(A))$ for all $A \in B(H)$. Therefore, for uniqueness we need to show that every extension $\hat{\rho}\colon B(H) \to \mathbb{C}$ of any pure state $\rho\colon D(H) \to \mathbb{C}$, satisfies $\hat{\rho}(A - \mathrm{diag}(A)) = 0$ for all $A \in B(H)$, or in other words, $\hat{\rho}$ vanishes on zero diagonal matrices. As an observation, if we consider only the diagonal projection operators $P \in D(H)$, which contain either 0 or 1 on their diagonals, then we have $\rho(P) \in \{0, 1\}$ for every pure state $\rho\colon D(H) \to \mathbb{C}$.

Lemma 2. *If $P \in D(H)$ is a diagonal projection and ρ is a pure state on $D(H)$, then $\rho(P) \in \{0, 1\}$.*

Proof. For the sake of contradiction, let us assume that $\rho(P) = \lambda \in (0, 1)$, which from linearity also implies $\rho(I - P) = 1 - \lambda$, where I is the identity element in $D(H)$. We can now consider the linear functionals, $\rho_1, \rho_2\colon D(H) \to \mathbb{C}$ defined as

$$\rho_1(M) \coloneqq \frac{1}{\lambda}\rho(PM), \quad \text{and} \quad \rho_2(M) \coloneqq \frac{1}{1-\lambda}\rho((I - P)M).$$

Observe that now we can write $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$, which contradicts the assumption that ρ is a pure state. □

As a consequence of Lemma 2, if we have any finite family of diagonal projections $P_1, \ldots, P_k \in D(H)$ for some $k \in \mathbb{N}$ that add up to the identity element, and we have a pure state extension $\hat{\rho}\colon B(H) \to \mathbb{C}$ then for exactly one $i \in [k]$, we will have $\rho(P_i) = 1$ with the rest zeros. Since these are diagonal projections, we have the same holding or their extensions, i.e., $\hat{\rho}(P_i) = 1$ with the rest zeros. Therefore it is relatively easy to understand the behavior of the extensions of pure states for diagonal projection operators. This allows us to introduce the notion of paving introduced by Anderson [And79a, And79b, And81], as we describe next.

1.1 Anderson's Paving Conjecture

Definition 3 (ϵ-paving)**.** An ϵ-paving of an operator $M \in B(H)$ is a finite collection of diagonal projections $\{P_i\}_{i=1}^k$ for some $k \in \mathbb{N}$ such that the projections add up to the identity, i.e., $\sum_{i=1}^k P_i = I$, and

$$\|P_i M P_i\|_2 \leq \epsilon \|M\|_2 \quad \forall\ i \in [k].$$

Then we can show the following lemma.

Lemma 4. *Let $\rho\colon D(H) \to \mathbb{C}$ be a pure state and $\hat{\rho}\colon B(H) \to \mathbb{C}$ be one of its extension to $B(H)$. If $M \in B(H)$ has an ϵ-paving, then $|\hat{\rho}(M)| \leq \epsilon\|M\|_2$.*

Proof. Let $\{P_i\}_{i=1}^k$ be the ϵ-paving of M. Then, from linearity of $\hat{\rho}$, we have

$$\begin{aligned}\hat{\rho}(M) &= \hat{\rho}(IMI)\\ &= \hat{\rho}\left(\sum_{i=1}^{k} P_i M \sum_{j=1}^{k} P_j\right)\\ &= \sum_{i,j=1}^{k} \hat{\rho}(P_i M P_j).\end{aligned} \tag{1.1}$$

From Lemma 2, we have that exactly one element in $\{P_i\}_{i=1}^k$, say P_ℓ satisfies $\hat{\rho}(P_\ell) = 1$. From Cauchy-Schwartz inequality for $\hat{\rho}$, we have

$$\begin{aligned}|\hat{\rho}(P_i M P_j)| &\leq \min\bigl\{\hat{\rho}(P_i^* P_i)\hat{\rho}(P_j^* M^* M P_j), \hat{\rho}(P_i^* M^* M P_i)\hat{\rho}(P_j^* P_j)\bigr\}\\ &= \min\bigl\{\hat{\rho}(P_i)\hat{\rho}(P_j^* M^* M P_j), \hat{\rho}(P_i^* M^* M P_i)\hat{\rho}(P_j)\bigr\} \qquad \Bigl(\because \{P_i\}_{i=1}^k \text{ are projections}\Bigr),\end{aligned} \tag{1.2}$$

which implies that all but the term $\hat{\rho}(P_\ell M P_\ell)$ are zero. Therefore,

$$\begin{aligned}\hat{\rho}(M) &= \hat{\rho}(P_\ell M P_\ell)\\ &\leq \|P_\ell M P_\ell\|_2\\ &\leq \epsilon \|M\|_2 \qquad \Bigl(\because \{P_i\}_{i=1}^k \text{ is an } \epsilon\text{-paving of } M\Bigr),\end{aligned} \tag{1.3}$$

which completes the proof. □

Lemma 4 implies the following important theorem.

Theorem 5. *For every $\epsilon > 0$, if every zero diagonal matrix $M \in B(H)$ has an ϵ-paving, then it implies a positive solution to the Kadison-Singer problem.*

Proof. Consider an operator $A \in B(H)$, then we have that for every $\epsilon > 0$, $A - \mathrm{diag}(A)$ has an ϵ-paving. Therefore, for all $\epsilon > 0$ and all extensions $\hat{\rho}\colon B(H) \to \mathbb{C}$ of $\rho\colon D(H) \to \mathbb{C}$, we have from Lemma 4 that

$$\begin{aligned} & \hat{\rho}(A - \mathrm{diag}(A)) \leq \epsilon \|A - \mathrm{diag}(A)\|_2 \quad \forall\, \epsilon > 0\\ \implies & \hat{\rho}(A - \mathrm{diag}(A)) = 0,\end{aligned} \tag{1.4}$$

which implies the uniqueness of the extension $\hat{\rho}$ implying a positive solution to the Kadison-Singer problem. □

Therefore, it is natural to conjecture the following two statements.

1.1.1 Infinite Paving Conjecture

Conjecture 6 (Infinite Paving Conjecture)**.** For every $\epsilon > 0$, every zero diagonal operator $M \in B(H)$ has an ϵ-paving.

A positive solution to Conjecture 6, which is a combinatorial question about partitioning matrices and has no mention of pure states, would readily imply a positive solution to the Kadison-Singer problem which is a question about extension of pure states on operator algebras. This conjecture has a finite version as well as we state next.

1.1.2 Finite Paving Conjecture

Conjecture 7 (Finite Paving Conjecture [And79a, And79b, And81])**.** There exist constants $\epsilon \in (0, 1)$ and $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, every zero diagonal matrix can be ϵ-paved with k diagonal projections.

Anderson used a limiting argument to prove that the Finite Paving Conjecture is equivalent to the infinite Paving conjecture. [Har13] thoroughly describe the reduction between the Kadison-Singer problem and the Paving Conjecture in detail. There are a series of statements in mathematics and engineering on partitioning matrices/vectors into 'balanced' parts, which were shown to be equivalent to the Finite Paving Conjecture [CEKP07]. Another such a statement is the one by Weaver, which we describe next.

1.2 Weaver's KS_r conjecture

The quadratic form associated with a set of vectors $\{v_i\}_{i=1}^m \subset \mathbb{C}^n$ is the bilinear form $\sum_{i=1}^m v_i v_i^*$, using which one can compute the second moment of vectors $\{v_i\}_{i=1}^m$ in a direction $u \in \mathbb{S}^{n-1}$ as

$$u^*\left(\sum_{i=1}^m v_i v_i^*\right)u = \sum_{i=1}^m |\langle u, v_i\rangle|^2.$$

The Weaver's KS_r conjecture for $r \in \mathbb{N} \setminus \{1\}$ is a question about partitioning an isotropic set of vectors for which the second moment is 1 is every direction $u \in \mathbb{S}^{n-1}$, i.e., $\sum_{i=1}^m v_i v_i^* = I_n$. We seek to partition the set into r sets of vectors such that none of the sets are degenerate, and each of them is close to have an isotropic bilinear form. Informally, we could be interested in finding a partition $\{T_j\}_{j=1}^r$ of $[m]$, such that $\sum_{i\in T_j} v_i v_i^* \approx I_n/r$, but such a partitioning, however, might not be possible always. We could have long vectors, which, if included in any set, would tend to create imbalances. Another reason for which this might not be possible is when we do not have enough vectors and we cannot avoid degenerate bilinear forms, for example consider any real orthogonal basis in $\mathbb{C}^n$. To avoid such obstacles, we can add an additional constraint of having 'short' vectors. Formally, Weaver's KS_r conjecture is stated below.

Conjecture 8 (Weaver's KS_r conjecture [Wea04])**.** There exist universal constants $\epsilon, \delta \in (0,1)$ such that for whenever $\{v_i\}_{i=1}^m \subset \mathbb{C}^n$ satisfies $\sum_{i=1}^m v_i v_i^* = I_n$ and $\|v_i\|_2 \le \delta$ for all $i \in [m]$, then there is a partition $\{T_j\}_{j=1}^r$ of $[m]$, such that

$$\left\|\sum_{i\in T_j} v_i v_i^*\right\|_2 \le 1-\epsilon \quad \forall\, j \in [r].$$

Note that the conjecture asks for constants ϵ and δ that are independent of n as well, otherwise for which we can have solutions for $\epsilon = 1/n$ using matroid theory. Weaver showed in the same work that a positive solution to the KS_r conjecture for any $2 \le r < \infty$ is equivalent to a positive solution to the Kadison-Singer problem.

2 Main result

Following the Weaver's KS_r conjecture 8, one can now ask whether there is a deterministic, or even a randomized procedure which can provide us with a partition satisfying the Weaver's conjecture. The problem is not so trivial, and we can understand the non-triviality from the below examples:

1. For $\delta > 0$ such that $1/\delta \in \mathbb{N}$, let $m = n/\delta$, and let the set of vectors $\{v_i\}_{i=1}^m$ be $\cup_{j=1}^n \{\delta^{1/2} e_j\}_{i=1}^{1/\delta}$, where $\{e_i\}_{i=1}^n$ is the real canonical basis on $\mathbb{C}^n$. This satisfies the conditions of Conjecture 8 since $\|v_i\|_2^2 = \delta$, and

$$\sum_{i=1}^m v_i v_i^* = \frac{1}{\delta}\sum_{j=1}^n \delta e_j e_j^* = I_n.$$

   Since the example is symmetric and dimension separable, it is not hard to find a partition. We could partition vectors along each dimension independently to obtain subsets of almost equal sizes which will be very close to being isotropic. This deterministic policy of partitioning the set of vectors had us to utilize the fact that the set of vectors can be split into n groups of one dimensional versions of the same problem. Such a partition is actually rare and we can see this by analyzing a sampling based method to achieve the same result. Let $r = 2$, and we randomly partition the vectors such that for each v_i for $i \in [m]$, i goes to T_1 with probability $1/2$, and in T_2 with the remaining probability $1/2$. Then, the probability with which there is no $j \in [n]$ such that all the vectors in the j-th dimension fall in some fixed partition is $\left(1 - 2^{-1/\delta}\right)^n$. Therefore, the probability that there exists a $j \in [n]$ such that all the vectors along the j-th direction are in the same set is $1 - \left(1 - 2^{-1/\delta}\right)^n$, which is large unless $\delta = \mathcal{O}(1/\log n)$. Therefore, a random partition is not a good policy since the condition for concentration scales with n.

2. For $\delta > 0$ such that $1/\delta \in \mathbb{N}$, let $m = n/\delta$, and let the set of vectors $\{v_i\}_{i=1}^m$ be randomly drawn from $\mathcal{N}(0, \delta I_n/n)$, where $\mathcal{N}$ denotes the Gaussian distribution taking a mean vector and a covariance matrix. This satisfies $\mathbb{E}\Big[\|v_i\|_2^2\Big] = \delta$. In fact, using [Bar05, Corollary 2.3] we have $\mathbb{P}\Big\{\|v_i\|_2^2 > (1+t)\delta\Big\} \le \mathrm{e}^{-t^2 n/4}$, which

using a union bound gives us that $\max_{i\in[m]}\|v_i\|_2^2 \le (1+o(1))\delta$ with probability at least $1-\exp(-n)$, as long as $m = \exp(o(n))$. The sampled vectors also satisfy $\mathbb{E}\left[\sum_{i\in[m]} v_i v_i^\top\right] = I_n$, and using [Ver10, Theorem 5.39] we have that the eigenvalues of $V \coloneqq \sum_{i\in[m]} v_i v_i^\top$ are in the interval $\left[(1-\mathcal{O}(\delta^{1/2}))^2, (1+\mathcal{O}(\delta^{1/2}))^2\right]$ with probability at least $1-\exp(-n)$. Thus, the normalized whitened vectors $\tilde{v}_i \coloneqq V^{-1/2} v_i$ for all $i\in[m]$ satisfy the conditions of Conjecture 8 with constants of similar order.

Since this is a randomized setup, coming up with a deterministic rule is not very immediate. However, a randomized partition works quite well. If we take a random equal partition T_1, T_2 of $[m]$ and define $V_j \coloneqq \sum_{i\in T_j} v_i v_i^\top$ for $j\in[2]$, then we can again use [Ver10, Theorem 5.39] to have that the eigenvalues of V_j lie in the interval $\left[(1-\mathcal{O}(\delta^{1/2}))^2/2, (1+\mathcal{O}(\delta^{1/2}))^2/2\right]$. Therefore, a random good partition is common but there is no deterministic rule to find one.

Therefore, this randomized method of partitioning the vectors works only if we have $\delta \le 1/\log n$. In order to have a dimension free result of the similar form, we might want to have

$$\sum_{i\in T_1} v_i v_i^\top \preceq \left(\frac{1}{2} + f(\epsilon)\right) I_n, \quad \text{and} \quad \sum_{i\in T_2} v_i v_i^\top \preceq \left(\frac{1}{2} + f(\epsilon)\right) I_n,$$

for some increasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. Here $\preceq$ denotes the standard partial ordering between positive semi-definite (PSD) operators. Instead of writing these two PSD inequalities, we can write this generally in a slightly different and compact form

$$\left\| \begin{bmatrix} \sum_{i\in T_1} v_i v_i^\top & 0 \\ 0 & \sum_{i\in T_2} v_i v_i^\top \end{bmatrix} \right\|_2 \le \frac{1}{2} + f(\epsilon). \tag{2.1}$$

The randomness is the sets T_1 and T_2 can be explicitly incorporated by introducing random vectors $w_i \sim \mathcal{U}\left(\begin{bmatrix} v_i \\ 0_n \end{bmatrix}, \begin{bmatrix} 0_n \\ v_i \end{bmatrix}\right)$ for all $i\in[m]$, where $\mathcal{U}$ denotes the independent uniform random distribution. Therefore, condition in Equation (2.1) can be written as

$$\left\| \sum_{i\in[m]} w_i w_i^\top \right\|_2 \le \frac{1}{2} + f(\epsilon), \tag{2.2}$$

which is a sum of independent rank-1 matrices. The expectation of this random matrix is $\mathbb{E}\left[\sum_{i\in[m]} w_i w_i^\top\right] = I_{2n}/2$. Using this formulation, we can use the Rudelson's inequality [Ver10, Corollary 5.28] to show that

$$\sum_{i\in T_j} v_i v_i^\top \preceq \left(\frac{1}{2} + \sqrt{\delta \log n}\right) I_n,$$

for all $j\in[2]$ with exponentially high probability in n, but as we see there is again a dimension factor $\sqrt{\log n}$ in this argument. Note that in order to argue about the existence of a partitioning, showing that we can obtain a dimension independent result with a positive probability, is sufficient. The authors in [MSS15] show that this can be shown and they prove the below result.

Theorem 9 ([MSS15])**.** *If $\delta > 0$, and $\{v_i\}_{i=1}^m$ for some $m\in\mathbb{N}$ are independent random vectors in $\mathbb{C}^n$ with finite support such that $\sum_{i=1}^m \mathbb{E}[v_i v_i^*] = I_n$, and $\mathbb{E}\left[\|v_i\|_2^2\right] \le \delta$ for all $i\in[m]$, then*

$$\mathbb{P}\left\{ \left\| \sum_{i=1}^m v_i v_i^* \right\|_2 \le \left(1+\sqrt{\delta}\right)^2 \right\} > 0.$$

The probability statement in Theorem 9 is strong since there is no dimension dependence, but weak as there is no high probability guarantee as it is true with only a positive probability. Using Theorem 9 we can therefore get the desired positive result for the KS_2 conjecture, that there exists a partition T_1, T_2 of $[m]$ such that

$$\sum_{i\in T_j} v_i v_i^\top \preceq \left(\frac{1}{2} + \mathcal{O}\left(\sqrt{\delta}\right)\right) I_n, \tag{2.3}$$

for all $j\in[2]$.

2.1 Arguments of the proof of Theorem 9

Before describing the proof of Theorem 9, we might first want to understand the possible strategies of proving the proof. Given a bunch of random vectors $\{v_i \in \mathbb{R}^n\}_{i=1}^m$ satisfying $\mathbb{E}\left[\|v_i\|_2^2\right] \leq \delta$ for all $i \in [m]$, and $\mathbb{E}\left[\sum_{i=1}^m v_i v_i^\top\right] = I_n$, we want to show that $A \coloneqq \sum_{i=1}^m v_i v_i^\top$ has a small spectral norm with a strictly positive probability. Standard approaches do not try to bound the spectral norm directly, but try to control it using a variational characterization of the spectral norm $\mathbb{E}\left[\sup_{x \in \mathbb{S}^{n-1}} x^\top A x\right]$; trace of higher powers of the matrix, $\mathbb{E}[\mathrm{Tr}[A^p]]$ for some $p \in \mathbb{N}$; or the moment generating function, $t \mapsto \mathbb{E}[\exp(tA)]$, etc. The standard methods which use these objects are used to show a high probability statement, however for this case since we already know that the statement corresponding to high probability is false, these methods do not provide much help. Also, these Chernoff bound based methods without any extra regularity or symmetry assumptions typically come with dimension dependence which we do not want.
The proof involves accessing the distribution of A by considering the expected characteristic polynomial $\mathbb{E}[\chi(A)]$ of A,

$$\mathbb{E}[\chi(A)](z) \coloneqq \mathbb{E}[\det(zI_n - A)]. \tag{2.4}$$

Before moving forward, let us again consider the examples from §2. For example 1, let $k = 1/\delta$, then the expected characteristic polynomial of the (diagonal) matrix $A = \sum_{i=1}^m b_i v_i v_i^\top$ where $b_i \sim \mathcal{U}\{0, 1\}$ for all $i \in [m]$, is

$$\mathbb{E}[\chi(A)](z) = \left(z - \frac{1}{2}\right)^n.$$

Observe that the expected characteristic polynomial is real rooted, which is not immediate since roots of a polynomial which is a sum of polynomials with real roots need not be real. Also observe that the largest root of this polynomial is $1/2 < 1$, whereas the roots of the polynomials which were averaged had a root at 1.
Now consider example 2. In this case, the expected characteristic polynomial is associated with the Laguerre Polynomial $\mathcal{L}_n^{m-n}$ whose maximum root is upper bounded by $\left(1 + \sqrt{\frac{n}{m}}\right)^2$ [Kra06]. Note that the expected characteristic polynomial is real rooted, and is a sum of polynomials whose roots are also bounded by the same quantity $\left(1 + \sqrt{\frac{n}{m}}\right)^2$.
The two examples therefore suggest a question whether the largest roots of $\chi(A)$ and $\mathbb{E}[\chi(A)]$ are related. The authors show that the expected characteristic polynomial always has real roots [MSS14, Theorem 4.1], and that the spectral norm is bounded above the largest root of the expected characteristic polynomial with a positive probability [MSS14, Theorem 1.7]. To understand this, let us first consider a basic question

Question 10. Given polynomials p_0, p_1, when are the roots of $\{p_i\}_{i=0}^1$ related to the roots of $\mathbb{E}_i[p_i]$?

The answer to Question 10 is not trivial, which we can see by a very simple example. Consider real rooted polynomials $p_0(x) = (x-1)^2$ and $p_1(x) = (x+1)^2$. Under the uniform distribution over $\{p_i\}_{i=0}^1$, the expected polynomial $\mathbb{E}_i[p_i](x) = x^2 + 1$ has imaginary roots. The main problem with analyzing the expected characteristic polynomial is that addition of the polynomials is a function of the coefficients, whereas we are interested in roots. Constructing coefficients from roots is an easy problem which we can do by simply multiplying the factors. However, constructing roots from coefficients is a hard problem as we know from Galois theory. However, if we consider polynomial whose roots are real and alternate, i.e., $\lambda_1(p_0) \leq \lambda_1(p_1) \leq \lambda_2(p_0) \leq \lambda_2(p_1)$, where λ_k maps a polynomial to its k-th largest roots, then under any distribution over $\{p_i\}_{i=0}^1$, the expected polynomial $\mathbb{E}_i[p_i](x)$ has real roots, and its roots lie between the roots of the individual polynomials, as we see from Figure 1.

2.1.1 Interlacing families

Definition 11 (Interlacing polynomials)**.** A real rooted polynomial g defined as $g(x) = \alpha_0 \prod_{i=1}^{n-1}(x - \alpha_i)$ interlaces a real rooted polynomial f defined as $f(x) = \beta_0 \prod_{i=1}^{n}(x - \beta_i)$, if $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_n$. Further, g strictly interlaces f if the inequalities are strict.

We can now prove the below lemma.

Lemma 12. *If p_0 and p_1 are monic polynomials which have a common interlacing, then there exists $i \in \{0, 1\}$ such that $\lambda_{\max}(p_i) \leq \lambda_{\max}(\mathbb{E}_i[p_i])$ under any distribution.*

The proof of Lemma 12 follows by considering a closed interval $[s, t] \subset \mathbb{R}$ such that $s = \min\{\lambda_{\max}(p_i)\}_{i=0}^1$ and $t = \max\{\lambda_{\max}(p_i)\}_{i=0}^1$, and using the mean value theorem. Since the random matrix A in Equation (2.4) is a sum

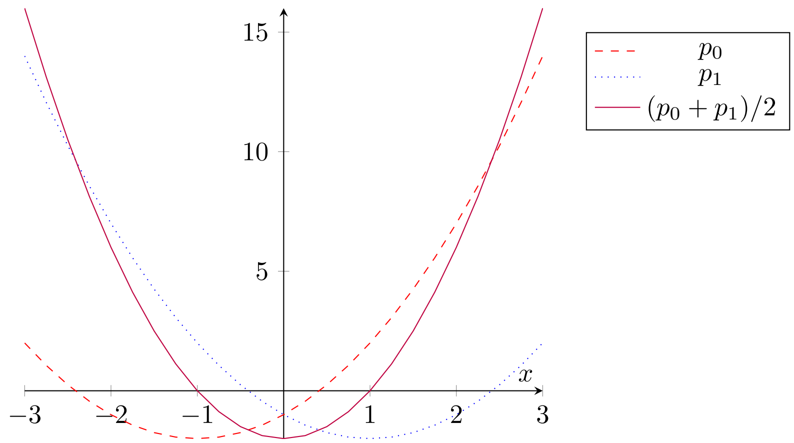

Figure 1: Interlacing polynomials and their expectation

of independent rank 1 random matrices, Cauchy's interlacing theorem tells us that when we add a symmetric rank 1 matrix to a symmetric matrix, then the eigenvalues of the original matrix interlaces the eigenvalues of the resulting sum. In general if we have a matrix $A \in \mathbb{R}^{n\times n}$ and a random vector $v \sim \mathcal{U}\{v_1, v_2\}$, then note that

$$\begin{aligned} p(x) \coloneqq \mathbb{E}\big[\chi\big(A + vv^\top\big)\big](x) &= \frac{1}{2}\chi\big(A + v_1v_1^\top\big)(x) + \frac{1}{2}\chi\big(A + v_2v_2^\top\big)(x) \\ &\coloneqq \frac{1}{2}p_0(x) + \frac{1}{2}p_1(x). \end{aligned} \tag{2.5}$$

Generally, we say that p forms an interlacing star with $\{p_i\}_{i=1}^m$ if p is a convex combination of $\{p_i\}_{i=1}^m$, each polynomial in $\{p_i\}_{i=1}^m$ is a real rooted degree d monic polynomial, and when $\{p_i\}_{i=1}^m$ has a common interlacer. Therefore, as a corollary of Lemma 12 we have the following.

Lemma 13. *If p forms an interlacing star with $\{p_i\}_{i=1}^m$, then there exists $i, j \in [m]$ such that*

$$\lambda_k(p_i) \le \lambda_k(p) \le \lambda_k(p_j).$$

The property of interlacing can be relaxed into the property of real-rootedness with the help of a folklore lemma.

Lemma 14. *Let $\{p_i\}_{i=1}^m$ be a collection of degree d monic polynomials. Then the following statements are equivalent.*

- *Every polynomial in the convex hull of $\{p_i\}_{i=1}^m$ has d real roots.*
- *The collection $\{p_i\}_{i=1}^m$ has a common interlacer.*

Therefore, if we replace the halves in Equation (2.5) with λ and $1-\lambda$ for $\lambda \in [0,1]$, and define

$$\bar{p}(x) = \lambda\chi(A + v_1v_1^\top)(x) + (1-\lambda)\chi(A + v_2v_2^\top)(x), \tag{2.6}$$

then there exists a common interlacer according to Lemma 14.

Recall that the central object which we want to analyze in Theorem 9 is $\lambda_1\big(\sum_{i=1}^m v_iv_i^\top\big)$, where the vectors $\{v_i\}_{i=1}^m$ are random vectors on a finite support. Therefore, if all the possible characteristic polynomials in the support had a common interlacer we would find a polynomial whose maximum roots would be strictly smaller than the maximum roots of the individual polynomials. This however, is too much to hope for. Instead, we can group the polynomials into smaller interlacing stars. We can then iterates over groups to construct an interlacing family. The main punchline of this construction is the following lemma.

Lemma 15. *Every interlacing family contains leaf nodes p_{leaf_1} and p_{leaf_2} such that*

$$\lambda_k(p_{\mathrm{leaf}_1}) \le \lambda_k(p_{\mathrm{top}}) \le \lambda_k(p_{\mathrm{leaf}_2}),$$

where p_{top} is the root node of the interlacing family.

As a consequence of this lemma, we can iteratively find polynomials which are strictly better by going down the tree of the interlacing family. Therefore, if there exists an interlacing family with $\left\{\chi\left(\sum_{i=1}^m v_i v_i^\top\right)\right\}_{\{v_i\}_{i=1}^m \in \operatorname{supp}\left(\{v_i\}_{i=1}^m\right)}$ as lead nodes, and $\mathbb{E}\left[\chi\left(\sum_{i=1}^m v_i v_i^\top\right)\right]$ as the top node, then

$$\min_{\{v_i\}_{i=1}^m \in \operatorname{supp}\left(\{v_i\}_{i=1}^m\right)} \left\| \sum_{i=1}^m v_i v_i^\top \right\|_2 \leq \lambda_1\left(\mathbb{E}\left[\chi\left(\sum_{i=1}^m v_i v_i^\top\right)\right]\right). \tag{2.7}$$

Therefore, we only now need to build this interlacing family with the above conditions to obtain the desired result. Therefore, let $V \coloneqq \sum_{i=1}^m v_i v_i^\top$ be the random matrix. Consider the following tree in Figure 2, where going down will be like revealing the value of each v_i in order. Here,

$$p_{s_1, s_2, \ldots, s_r} \coloneqq \mathop{\mathbb{E}}_{v_{r+1}, \ldots, v_m}[\chi(V) \mid v_1 = s_1, \ldots, v_r = s_r]. \tag{2.8}$$

In particular, the siblings at depth $t \in [m]$ differ only in the value of v_t, i.e., any collection of children are going to be the choice of some new vector after we already know all the vectors that the node has.

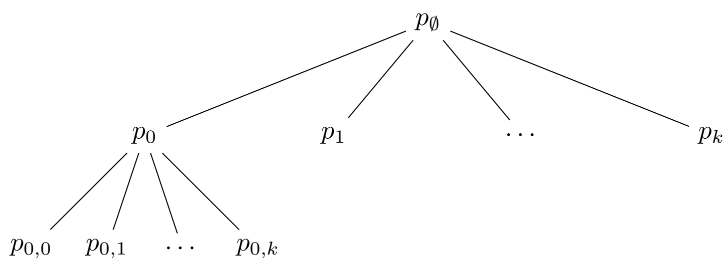

Figure 2: Building an interlacing family

The polynomials in the interlacing family undertake a special form, called the mixed characteristic polynomials which we will discuss in the next section.

2.1.2 Mixed Characteristic Polynomials

We shall now develop useful expressions for the expected characteristic polynomial and show that these polynomials are real-rooted, which will be crucial for interlacing method.

For any invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$, and vectors $u, v \in \mathbb{R}^n$, from the properties of the determinant we have $\det(A + uv^*) = \det(A)\left(1 + v^* A^{-1} u\right)$. Also, from the Jacobian formula for the derivatives of determinants, we have for any $B \in \mathbb{R}^{n \times n}$, $\partial_t \det(A + tB)\big|_{t=0} = \det(A)\operatorname{Tr}\left[A^{-1}B\right]$, therefore

$$\begin{aligned}
\mathbb{E}[\det(A - vv^*)] &= \mathbb{E}\left[\det(A)\left(1 - v^* A^{-1} v\right)\right] \\
&= \mathbb{E}\left[\det(A)\left(1 - \operatorname{Tr}\left[A^{-1} vv^*\right]\right)\right] \\
&= \det(A)\left(1 - \mathbb{E}\left[\operatorname{Tr}\left[A^{-1} vv^*\right]\right]\right) \\
&= \det(A + t\,\mathbb{E}[vv^*])\big|_{t=0} - \det(A)\operatorname{Tr}\left[A^{-1}\,\mathbb{E}[vv^*]\right] \\
&= (1 - \partial_t)\det(A + t\,\mathbb{E}[vv^*])\big|_{t=0}.
\end{aligned} \tag{2.9}$$

Let $A_i \coloneqq \mathbb{E}[v_i v_i^*]$, then we have the useful theorem

Theorem 16.

$$\mathbb{E}\left[\chi\left(\sum_{i=1}^m v_i v_i^*\right)\right](x) = \left[\prod_{i=1}^m (1 - \partial_{z_i})\right] \det\left(x I_n + \sum_{i=1}^m z_i A_i\right)\Bigg|_{\{z_i = 0\}_{i=1}^m}. \tag{2.10}$$

Proof. For any $M \succ 0_{n \times n}$, let us define

$$a_k(M) \coloneqq \mathbb{E}\left[\det\left(M - \sum_{i=1}^k v_i v_i^*\right)\right]$$

$$b_k(M) := \left[\prod_{i=1}^{k}(1-\partial_{z_i})\right]\det\left(M+\sum_{i=1}^{k} z_i A_i\right)\Bigg|_{\{z_i=0\}_{i=1}^{k}},$$

Then we will prove by induction that $a_k(M) = b_k(M)$. Note the base case that

$$a_0(x) = \mathbb{E}[\det(M)] = \det(M) = b_0(x). \tag{2.11}$$

Then by strong induction, assume that $a_i(M) = b_i(M)$ for all $i \in [k-1]$. Then using Equation (2.9) we get

$$\begin{aligned}
a_k(M) &= \mathbb{E}\left[\det\left(M - \sum_{i=1}^{k} v_i v_i^*\right)\right] \\
&= \mathop{\mathbb{E}}_{\{v_j\}_{j=1}^{k-1}}\left[\mathbb{E}\left[\det\left(M - \sum_{i=1}^{k} v_i v_i^* - v_k v_k^*\right)\right]\right] \\
&= \mathop{\mathbb{E}}_{\{v_j\}_{j=1}^{k-1}}\left[(1-\partial_{z_k})\det\left(M - \sum_{i=1}^{k-1} v_i v_i^* + z_k A_k\right)\right]\Bigg|_{z_k=0} \\
&= (1-\partial_{z_k})\left(\prod_{i=1}^{k-1}(1-\partial_{z_i})\right)\det\left(M+\sum_{i=1}^{k} z_i A_i\right)\Bigg|_{\{z_i=0\}_{i=1}^{k-1}}\Bigg|_{z_k=0} \qquad \text{(Using the induction hypothesis)} \\
&= \left(\prod_{i=1}^{k}(1-\partial_{z_i})\right)\det\left(M+\sum_{i=1}^{k} z_i A_i\right)\Bigg|_{\{z_i=0\}_{i=1}^{k}} \\
&= b_k(M).
\end{aligned} \tag{2.12}$$

Hence, $a_k(M) = b_k(M)$ for all $M \succ 0_{d\times d}$. In particular, $a_m(xI_n) = b_m(xI)$ for $x > 0$. Since a_m and b_m are finite degree polynomials, so equality on any interval implies equality everywhere proving the theorem. □

Note that Theorem 16 implies that the expected characteristic polynomial only depends on the expectation of the outer products of the rank-1 matrices $\{v_i v_i^*\}_{i=1}^{m}$. This polynomial is called the mixed characteristic polynomial $\mu[A_1, \ldots, A_m]$ of positive definite matrices $\{A_i\}_{i=1}^{m}$.

We will now show that each polynomial defined in Equation (2.8) is a mixed characteristic polynomial. To see this, observe that the leaf polynomials is nothing but $\chi(V) = \mu\big[v_1 v_1^\top, \ldots, v_m v_m^\top\big]$. The top polynomial $\mathbb{E}[\chi(V)] = \mu[A_1, \ldots, A_m]$, and any polynomial in the k-th level are nothing but $\mu\big[v_i v_1^\top, \ldots, v_k v_k^\top, A_{k+1}, \ldots, A_m\big]$. In fact, something more is true, which is that every convex combination of siblings in the tree is again a mixed characteristic polynomial.

Therefore, from Lemma 14 it suffices to show that all the mixed characteristic polynomials are real rooted. This question is rather not straightforward since we don't have a good handle on the roots of the polynomials at the first place. Similar question have been studied in the early 20^{th} century in the context of the Riemann hypothesis, which is if there exist transformations which preserve real-rootedness. This brings us to the literature of 'stable polynomials' which we are going to brief next.

2.1.3 Real Stable Polynomials

Definition 17 ($\mathbb{D}$-stable polynomial)**.** If $\mathbb{D} \subseteq \mathbb{C}$ is a domain, then a multivariate polynomial is called $\mathbb{D}$-stable if it is either identically zero, or it is never zero in $\mathbb{D} \times \cdots \times \mathbb{D}$.

We are going to be interested in the case when $\mathbb{D} = \mathbb{H}$, where $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z_i) > 0\}$. We will call $\mathbb{H}$-stable polynomials simply, stable polynomials. In addition, if all the coefficients of the stable polynomial are real, then we call it a real stable polynomial.
We have the following useful and important properties of real stable polynomials:

1. Univariate polynomials are real rooted if and only if they are real stable.
2. If $p(z_1, z_2, \ldots, z_n)$ is a real stable polynomial, then for any $a \in \mathbb{R}$, $p(a, z_2, \ldots, z_n)$ is also real stable.
3. If $p(z_1, z_2, \ldots, z_n)$ is a real stable polynomial, then $p(x, x, z_3, \ldots, z_n)$ is also real stable.

Furthermore we have two useful lemma.

Lemma 18. *If $\{A_i\}_{i=1}^m$ are Hermitian positive semi-definite matrices and $\{x_i\}_{i=1}^m$ are variable, then*

$$p\colon (x_i)_{i=1}^m \mapsto \det\left[\sum_{i=1}^m x_i A_i\right]$$

is a real stable polynomial.

The proof of Lemma 18 can be found in [SW20, Proposition 2.6].

Lemma 19. *If $p \in \mathbb{R}[z_1, \ldots, z_m]$ is a real stable polynomial, then so is*

$$(1 - \lambda\,\partial_{z_1})p,$$

for any $\lambda \in \mathbb{R}$.

Getting back to our claim that all mixed characteristic polynomials are real rooted, the claim is immediate using Lemma 18 and Lemma 19 since a mixed characteristic polynomial is a composition of the real stability preserving operations in these lemmas.

Combining these arguments we establish Equation (2.7). This bound alone is of not much good since we do not have much control on the right hand side term $\lambda_1\big(\mathbb{E}\big[\chi\big(\sum_{i=1}^m v_i v_i^\top\big)\big]\big)$. Illuminating on this term we need to introduce a theory of the roots of multivariate polynomials.

2.1.4 Roots of multivariate polynomials

Rather than having roots that are just points, multivariate polynomials have zero surfaces called varieties. A degree d real stable polynomial has d varieties which partition $\mathbb{R}^n$ into $d+1$ components. Out of all the components, there are two which are special ones - positive and the negative component. The positive component is the one which contains a translation of the positive orthant of $\mathbb{R}^n$, and the negative component is the one which contains a translation of the negative orthant of $\mathbb{R}^n$. Both these special orthants are in fact convex, as shown in Figure 3.

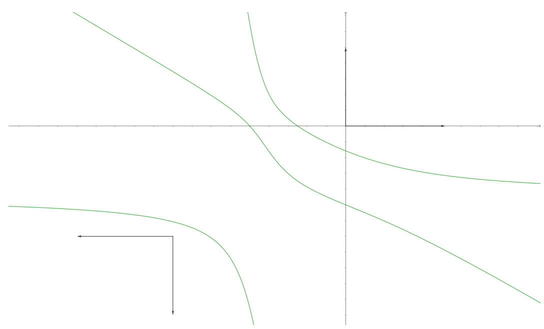

Figure 3: Varieties of the polynomial $\mathbb{R}^2 \ni (x, y) \mapsto 4 + 12x + 8x^2 + 17y + 29xy + 9x^2y + 14y^2 + 13xy^2 + y^3 \in \mathbb{R}$ with the positive and negative component regions.

If we restrict the polynomial along a hyperplane, we obtain a univariate polynomial, for example, setting $y = x$ or $y = -2x$ in the multivariate polynomial in Figure 3 yields us two univariate polynomials both of which are real rooted. Therefore, considering again the expression in Theorem 16, if we have $\sum_{i=1}^m A_i = I_n$, then defining new variables $\{y_i \coloneqq z_i + x\}_{i=1}^m$, we get

$$\mu[A_1, \ldots, A_m](x) = \left[\prod_{i=1}^m (1 - \partial_{y_i})\right] \det\left(\sum_{i=1}^m y_i A_i\right)\Bigg|_{\{y_i = x\}_{i=1}^m}. \tag{2.13}$$

Recall that now our problem is to bound $\lambda_1\big(\mathbb{E}\big[\chi\big(\sum_{i=1}^m v_i v_i^\top\big)\big]\big)$ in the case that $\mathbb{E}\big[\sum_{i=1}^m v_i v_i^\top\big] = I_n$ and $\mathbb{E}\Big[\|v_i\|_2^2\Big] \le \delta$. Therefore this is equivalent to finding a real number $t \in \mathbb{R}$ such that the vector $t1_n$ stays on the positive component of the mixed characteristic polynomial in Equation (2.13), whenever $\sum_{i=1}^m A_i = I_n$ and $\mathrm{Tr}[A_i] \le \delta$ for all $i \in [m]$

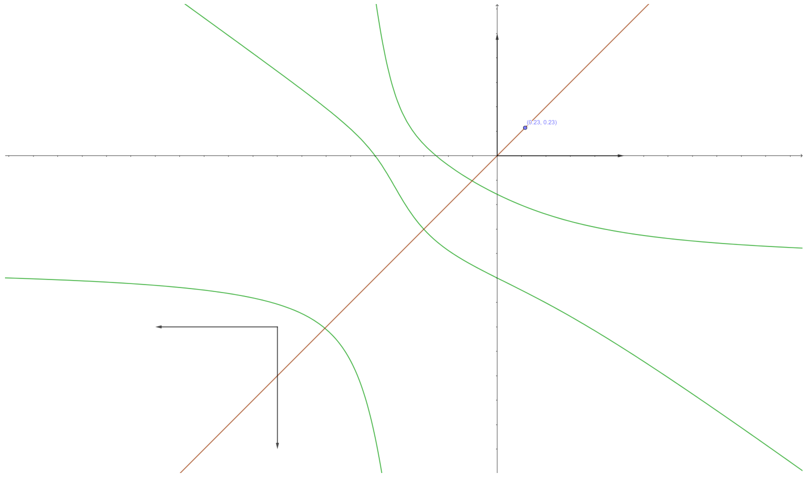

Figure 4: Restricting the polynomial $\mathbb{R}^2 \ni (x, y) \mapsto 4 + 12x + 8x^2 + 17y + 29xy + 9x^2y + 14y^2 + 13xy^2 + y^3 \in \mathbb{R}$ on $y = x$. We want to find a point (t, t) which stays in a positive component of the polynomial.

(refer Figure 4). Therefore, the main idea is to apply the differential operators in the definition of the polynomial iteratively to see what happens to the positive component of the polynomial. In order to measure this effect, we can use 'barrier functions'. Barrier function of a polynomial p is the derivative of the logarithm of the polynomial.

$$\Phi_p^i(z_1, \ldots, z_m) \coloneqq \frac{\partial}{\partial z_i} \log p(z_1, \ldots, z_m).$$

This Barrier function is a generalization of what was first defined in [BSS12] which for a univariate polynomial is defined as

$$\phi_p(x) \coloneqq \frac{\partial}{\partial x} \log p(x) = \frac{p'(x)}{p(x)} = \sum_{i=1} \frac{1}{x - r_i},$$

where $\{r_i\}_i$ are the roots of the polynomial p. In particular, each of $\left\{\Phi_p^i\right\}_{i=1}^m$ is monotone non-increasing convex on the open positive component. This can be thought to measure the amount of 'cushion' in a direction. Note that

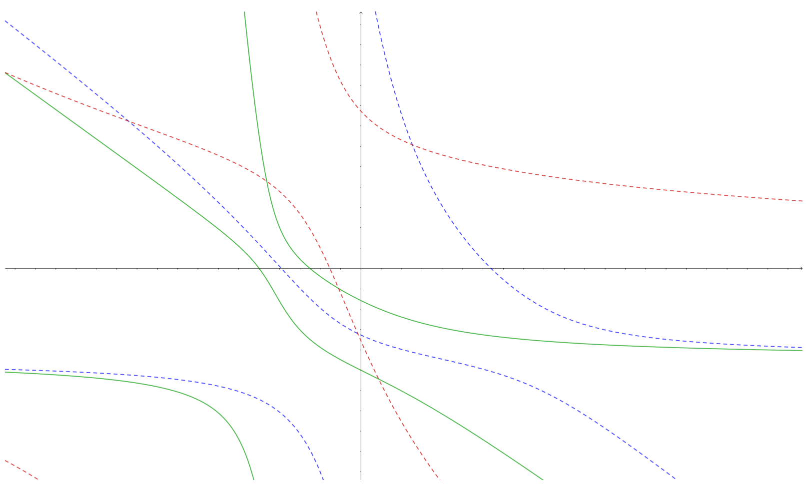

Figure 5: Varieties of the polynomial $p(x, y) = 4 + 12x + 8x^2 + 17y + 29xy + 9x^2y + 14y^2 + 13xy^2 + y^3$ (green), $\Phi_p^1 - 1$ (blue dashed) and $\Phi_p^2 - 1$ (red dashed).

the variety of $\Phi_p^i - s$ for $s \in \mathbb{R}$ and $i \in [m]$ in the positive component of p drifts towards the positive component of p as s increases. If we consider a single operator $(1 - \partial_i)$ for the i-th direction for any $i \in [n]$, we note that it does two things:

1. Shifts the entire region in the i-th direction. For example, in the Figure 4, for any translated positive orthant of $\mathbb{R}^2$ in the positive component of p, there exists another translation (only in the i-th direction) of the positive orthant in the positive component of $\Phi_p^i - 1$.
2. Causes the region to flatten inwards in all directions. For example, in the Figure 4, the variety of $\Phi_p^i - 1$ for any $i \in [2]$ contained in the positive component of p is 'smoother' than the variety of p corresponding to the positive component of p.

This brings us to the following lemma.

Lemma 20 (Monotonicity of Φ_p^i)**.** *If p is a real stable polynomial, and $z \in \mathbb{R}^n$ is in the positive component of p, and $\Phi_p^i(z) < 1$ for some $i \in [n]$, then z is in the positive component of $(1 - \partial_i)p$.*

Lemma 20 just shows what happens after we apply a single operator $(1 - \partial_j)$ for some $j \in [n]$, it is not strong enough for an inductive application of all such operators in the definition of the mixed characteristic polynomial in Equation (2.13) as the application of a single such operator typically increases all the barrier functions simultaneously. As we note in points 1 and 2, the effect of a single such operator causes a shift along the same direction and a flattening away from the origin. To remedy this, we need to translate our upper bounds in the same direction as well.

Lemma 21. *If p is a real stable polynomial, and $z \in \mathbb{R}^n$ is in the positive component of p and $\Phi_p^j(z) \leq 1 - \frac{1}{\eta}$ for some $\eta \in \mathbb{R}_{++}$ and $j \in [n]$, then for all $i \in [n]$ we have $\Phi^i_{(1-\partial_j)p}(z + \eta e_j) \leq \Phi_p^i(z)$.*

The proof of Lemma 21 follows from the convexity properties of the barrier function and is formally derived in [MSS15]. As a consequence, and as shown in Figure 6, the shifted point $z' = z + \eta e_2$ recovers the cushion in all

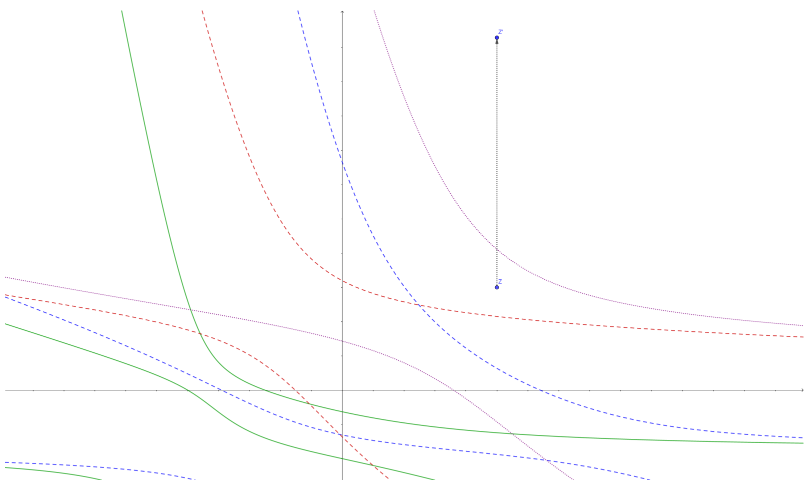

Figure 6: Varieties of the polynomial $p(x, y) = 4 + 12x + 8x^2 + 17y + 29xy + 9x^2y + 14y^2 + 13xy^2 + y^3$ (green), $\Phi_p^1(x, y) - 1$ (blue dashed), $\Phi_p^2(x, y) - 1$ (red dashed), and $\Phi^1_{(1-\partial_2)p} - 1$ (purple dotted). The point z satisfies $\Phi_p^2(z) = 1 - \eta$ for $\eta = 353/1287$, and $z' = z + \eta e_2$.

the directions. Iterating on the application of all the differential operators, we obtain the following theorem.

Theorem 22. *The point $\left(1 + \sqrt{\delta}\right)^2 1_m$ is in the positive component of*

$$z \mapsto \left[\prod_{i=1}^{m} (1 - \partial_{z_i})\right] \det\left(\sum_{i=1}^{m} z_i A_i\right),$$

where $\sum_{i=1}^m A_i = I_n$ and $\mathrm{Tr}[A_i] \leq \delta$.

To start with, we look at the initial real stable polynomial $Q_0(z) := \det[\sum_{i=1}^m z_i A_i]$ and a point $w_0 = t1_m$ for some $t \in \mathbb{R}_{++}$ (to be optimized later) which is in the positive component of Q_0. Now note that for all $i \in [n]$,

$$\Phi^i_{Q_0}(w_0) = \mathrm{Tr}\left[\left(\sum_{j=1}^{m} tA_j\right)^{-1} A_i\right] = \frac{\mathrm{Tr}[A_i]}{t} \leq \frac{\delta}{t}.$$

This satisfies Lemma 21 in all directions for any $\eta \geq \frac{1}{1-\delta/t}$. Therefore, when we apply the first operator $(1-\partial_{z_1})$, we also move by η in the direction of e_1. From Lemma 21, $w_1 := w_0 + \eta e_1$ is in the positive component of $Q_1 := (1-\partial_{z_1})Q_0$ and we regain the cushion in each direction $\{e_j\}_{j=2}^m$. Therefore, continuing on the same argument on all other operators and directions, we get that $w_m := w_0 + \eta \sum_{i=1}^m e_i = (\eta + t)1_m$ is in the positive component of

$$z \mapsto Q_m(z) := \left[\prod_{i=1}^m (1-\partial_{z_i})\right] Q_0(z) = \left[\prod_{i=1}^m (1-\partial_{z_i})\right] \det\left(\sum_{i=1}^m z_i A_i\right).$$

Since η is a function of t and δ, optimizing $\eta + t$ for the best choice of t, we get $\eta + t$ can be as low as $\left(1+\sqrt{\delta}\right)^2$ as required. Winding back, we obtain that the maximum roots of the expected characteristic polynomial is upper bounded by $\left(1+\sqrt{\delta}\right)^2$, which is dimension independent. The existence of such a solution therefore provides us a positive probability to the event in Theorem 9 as desired.

2.2 Nice consequences and improvements

A beautiful consequence of the Kadison Singer solution is the one provided by [AW14, Theorem 1.4].

Theorem 23. *If $\left\{w_i \in \mathbb{C}^d\right\}_{i=1}^m$ such that $\sum_{i=1}^m w_i w_i^* \leq I_d$ and $\|w_i\|_2^2 \leq \delta$ for all $i \in [m]$, then for any collection of real numbers $0 \leq t_1, \ldots, t_m \leq 1$, there exists a set $S \subseteq [m]$ such that*

$$\left\| \sum_{i \in S} w_i w_i^* - \sum_{i=1}^m t_i w_i w_i^* \right\|_2 = \mathcal{O}\left(\delta^{1/8}\right).$$

Theorem 23 essentially says that the image of $[0,1]^n$ under the linear map $\psi\colon (t_i)_{i=1}^m \mapsto \sum_{i=1}^m t_i w_i w_i^*$, is well approximated by the image of $\{0,1\}^n$. This screams out ideas in semi-definite programming relaxation where we are trying to show possibly that when we want solve a problem over the discrete domain, we can assert that something in the continuous relaxed domain is actually close to the solution in the discrete domain.

Along with many, some applications of the result by Marcus et al. worth mentioning are improved bounds on asymmetric traveling salesman problem [AG14], improved restricted invertibility property [NY17], existence of Ramanujan graphs of all degree and all sizes [MSS14], along with constructions of such solutions [Coh16b, Coh16a].

References

[AG14] Nima Anari and Shayan Oveis Gharan. The kadison-singer problem for strongly rayleigh measures and applications to asymmetric tsp. *arXiv preprint arXiv:1412.1143*, 2014.

[And79a] Joel Anderson. Extensions, restrictions, and representations of states on C^*-algebras. *Transactions of the American Mathematical Society*, 249(2):303–329, 1979.

[And79b] Joel Anderson. Extreme points in sets of positive linear maps on $B(H)$. *Journal of Functional Analysis*, 31(2):195–217, 1979.

[And81] Joel Anderson. A conjecture concerning the pure states of $B(H)$ and a related theorem. In *Topics in modern operator theory*, pages 27–43. Springer, 1981.

[AW14] Charles Akemann and Nik Weaver. A lyapunov-type theorem from kadison–singer. *Bulletin of the London Mathematical Society*, 46(3):517–524, 2014.

[Bar05] Alexander Barvinok. Math 710: Measure concentration. *Lecture notes*, 2005.

[BSS12] Joshua Batson, Daniel A Spielman, and Nikhil Srivastava. Twice-ramanujan sparsifiers. *SIAM Journal on Computing*, 41(6):1704–1721, 2012.

[CEKP07] Peter G Casazza, Dan Edidin, Deepti Kalra, and Vern I Paulsen. Projections and the kadison-singer problem. *arXiv preprint math/0701450*, 2007.

[Coh16a] Michael Cohen. *Improved Spectral Sparsification and Kadison-Singer for Sums of Higher-rank Matrices*. PhD thesis, University of British Columbia, 2016.

[Coh16b] Michael B Cohen. Ramanujan graphs in polynomial time. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 276–281. IEEE, 2016.

[Har13] Nicholas Harvey. An introduction to the kadison-singer problem and the paving conjecture. *preprint available at https://www. cs. ubc. ca/nickhar/Publications/KS/KS. pdf*, 2013.

[Kla15] Erica Klarreich. 'outsiders' crack 50-year-old math problem. https://www.quantamagazine.org/computer-scientists-solve-kadison-singer-problem-20151124, Nov 2015.

[Kra06] Ilia Krasikov. On extreme zeros of classical orthogonal polynomials. *Journal of computational and applied mathematics*, 193(1):168–182, 2006.

[KS59] Richard V Kadison and Isadore M Singer. Extensions of pure states. *American journal of mathematics*, 81(2):383–400, 1959.

[MS17] Adam W Marcus and Nikhil Srivastava. The solution of the kadison-singer problem. *arXiv preprint arXiv:1712.08874*, 2017.

[MSS14] Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. Ramanujan graphs and the solution of the kadison-singer problem. *arXiv preprint arXiv:1408.4421*, 2014.

[MSS15] Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. Interlacing families ii: Mixed characteristic polynomials and the kadison—singer problem. *Annals of Mathematics*, pages 327–350, 2015.

[NY17] Assaf Naor and Pierre Youssef. Restricted invertibility revisited. In *A journey through discrete mathematics*, pages 657–691. Springer, 2017.

[SW20] Raghav Somani and Mitchell Wortsman. A survey on strongly rayleigh measures and their mixing time analysis, 2020. https://raghavsomani.github.io/files/SRmixing.pdf.

[Ver10] Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.

[Wea04] Nik Weaver. The kadison–singer problem in discrepancy theory. *Discrete mathematics*, 278(1-3):227–239, 2004.

A Definitions

Definition 24 (Weak operator topology)**.** The weak operator topology is the weakest topology on the set of bounded operators on a Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, such that the functionals that map an operator T to $\langle Tx, y \rangle_H$ are continuous for all $x, y \in H$.

Definition 25 (C^*-algebra)**.** A C^*-algebra $\mathcal{A}$ is an associative unital algebra which is a $\mathbb{C}$-Banach space equipped with an involution $^* \colon \mathcal{A} \to \mathcal{A}$ and a norm $\|\cdot\| \colon \mathcal{A} \to \mathbb{R}$ satisfying for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$,

1. $(x^*)^* = x$,
2. $(x + y)^* = x^* + y^*$,
3. $(xy)^* = y^* x^*$,
4. $(\lambda x)^* = \bar{\lambda} x^*$, and
5. $\|x^* x\| = \|x\| \|x^*\| = \|x x^*\| = \|x\|^2$.

Definition 26 (von Neumann algebra)**.** The von Neumann algebra is a C^*-algebra of bounded operators on a Hilbert space that is closed in the weak operator topology that contains the identity.

Definition 27 (w^*-topology)**.** If (X, τ) is a topological vector space with the dual X^*, then the w^*-topology on X^* can be defined as the coarsest topology under which every element $x \in X$ corresponds to a continuous map on X^*.